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On The Structure Constants Of Leibniz Algebras

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Abstract: Leibniz algebras are generalization of Lie algebras. As an immediate consequence, every Lie algebras are Leibniz algebras. In literature, there are many papers on finite dimensional Leibniz algebras.In this note, our main goal is to focus on the structure constants of Leibniz algebras and to give some properties of the structure constants of Leibniz algebras. In particular, some conditions are investigated for non-Lie Leibniz algebras. Moreover, some examples on structure constants of non-Lie Leibniz algebras are given and the structure constants of Leibniz kernel of a Leibniz algebra is investigated.

Keywords: Leibniz algebra, Lie algebra, structure constants, Leibniz kernel ideal.

I. INTRODUCTION

Recall that Lie algebra L over a field F, with a bilinear map, the Lie product $[\,,]: L \times L \to L$ defined by $(x, y) \to [x, y]$, satisfies the properties of anti-commutative and Jacobi identity (see [Erdmann and Wildon, 2006; Jacobson, 1979]). Suppose that L is an algebra over a field F with binary operations $+$ and $[\,,]$, then L is called a (left) Leibniz algebra if it satisfies the Leibniz identity

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]] \qquad (1)$$

for all $x, y, z \in L$. Throughout of this paper, we suppose all Leibniz algebras as a left Leibniz algebra. Every Lie algebra is a Leibniz algebra. Conversely, the Leibniz algebra L which satisfies that $[x, x] = 0$ for every $x \in L$ is a Lie algebra (for more details see [Bloh, 1965; Loday, 1993]).

By $Leib(L)$, we denote the subspace generated by the elements $[x, x]$, for some $x \in L$. This subspace is an ideal of L and it is said to be the Leibniz kernel of L. Since for $[x, x] \in Leib(L)$ and $y \in L$,

$$[[x, x], y] = [x, [x, y]] - [x, [x, y]] = 0. \qquad (2)$$

The Leibniz kernel of L is an abelian Leibniz algebra. If L is a non-Lie Leibniz algebra, then $Leib(L) \neq 0$ and this follows that there exists an element $0 \neq [x, x] \in Leib(L)$, for $x \in L$.
If L is a Leibniz algebra over a field F with basis $\{x_1, x_2, \dots, x_n\}$, then all elements in L can be determined by the products $[x_i, x_j]$. Moreover, each product $[x_i, x_j]$ is written as a linear combination of the elements of basis as the following

$$[x_i, x_j] = \sum_{k=1}^{n} c_{ij}^k \, x_k, \qquad (3)$$

where for $1 \leq i, j, k \leq n$, c_{ij}^k are scalars in F. The c_{ij}^k are called the structure constants of L depending on the choice of basis of L.

Here, the next question that arises naturally is as follows:

Question 1. Let L be a Leibniz algebra with basis $\{x_1, x_2, \dots, x_n\}$. What condition does the Leibniz identity impose on the structure constants c_{ij}^k?

II. MAİN RESULTS

In this section firstly we observe the structure constants of Leibniz algebras and then we give some results studied on the structure constants of non-Lie Leibniz algebras. We give the answer of Question 1. affirmatively, namely, we prove the following result.

Theorem 1. Let L be an algebra over a field F with basis $\{x_1, x_2, \dots, x_n\}$. The algebra L satisfies the Leibniz identity if and only if

$$\sum_{m=1}^{n} c_{ij}^k \, c_{mk}^s - \sum_{p=1}^{n} c_{jk}^p \, c_{ip}^s + \sum_{l=1}^{n} c_{ik}^l \, c_{jl}^s = 0.$$

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Proof. Assume that L is a Leibniz algebra. Then, from (1), we infer that

$$\left[[x_i, x_j], x_k\right] - \left[x_i, [x_j, x_k]\right] + \left[x_j, [x_i, x_k]\right] = 0.$$

By using (3), a straightforward calculation shows that

$$\sum_{s=1}^{n}\left(\sum_{m=1}^{n} c_{ij}^{m} c_{mk}^{s} - \sum_{p=1}^{n} c_{jk}^{p} c_{ip}^{s} + \sum_{l=1}^{n} c_{ik}^{l} c_{jl}^{s}\right) x_s = 0.$$

Accordingly, for all $1 \le s \le n$, x_s are linearly independent. Hence,

$$\sum_{m=1}^{n} c_{ij}^{m} c_{mk}^{s} - \sum_{p=1}^{n} c_{jk}^{p} c_{ip}^{s} + \sum_{l=1}^{n} c_{ik}^{l} c_{jl}^{s} = 0.$$

Conversely, it is not hard to show it.

Now we investigate the conditions when L is a non-Lie Leibniz algebra.

Theorem 2. Let L be a non-Lie Leibniz algebra with basis $\{x_1, x_2, \dots, x_n\}$. Then the structure constants c_{ij}^{k} of L provide the following properties:

(i)for some i, m, $c_{ii}^{m} \neq 0$,

(ii) for some i, j and m, $c_{ij}^{m} + c_{ji}^{m} \neq 0$,

(iii) $\sum_{m=1}^{n} c_{ij}^{m} c_{mk}^{s} - \sum_{p=1}^{n} c_{jk}^{p} c_{ip}^{s} + \sum_{l=1}^{n} c_{ik}^{l} c_{jl}^{s} = 0$

for all i, j, k, l, m, p and s.

Proof. (i) Recall that if L is a Lie algebra, then for each i, $[x_i, x_i] = 0$ and from (3), we have

$$[x_i, x_i] = \sum_{m=1}^{n} c_{ii}^{m} x_m.$$

Since for all $1 \le m \le n$, x_m are linearly independent, $c_{ii}^{m} = 0$. As a summarise, $[x_i, x_j] = 0$ if and only if $c_{ii}^{m} = 0$. Because of our assumption, for some i, m, $c_{ii}^{m} \neq 0$.

(ii) If L is a non-Lie Leibniz algebra, then for some $1 \le i, j \le n$, we have $[x_i, x_j] \neq -[x_j, x_i]$. Hence, $[x_i, x_j] + [x_j, x_i] \neq 0$. By using (3), we obtain

$$[x_i, x_j] + [x_j, x_i] = \sum_{m=1}^{n} c_{ij}^{m} x_m + \sum_{m=1}^{n} c_{ji}^{m} x_m = \sum_{m=1}^{n} (c_{ij}^{m} + c_{ji}^{m}) x_m.$$

This means that for some $1 \le i, j, m \le n$, $c_{ij}^{m} + c_{ji}^{m} \neq 0$.

(iii) It is clear to see this fact from Theorem 1.

Now we examine the structure constants of the Leibniz kernel of L. Say $K = Leib(L)$. First assume that for each element $x \in L$, $[x, x] \in K$. Since the element x can be expressed as a linear combination of x_i for $1 \le i \le n$, we have

$$\begin{aligned}[x, x] &= \left[\sum \alpha_i x_i, \sum \alpha_i x_i\right] \\ &= \sum_{m,i=1}^{n} \alpha_i^2 c_{ii}^{m} x_m + \sum_{m,i,j=1 (i \neq j)}^{n} c_{ij}^{m} x_m.\end{aligned}$$

We know that if L is a non-Lie Leibniz algebra, then there exists an element $x \in L$ such that $0 \neq [x, x] \in K$. Thus,

$$0 \neq [x, x] = \sum_{m,i=1}^{n} \alpha_i^2 c_{ii}^{m} x_m + \sum_{m,i,j=1 (i \neq j)}^{n} c_{ij}^{m} x_m.$$

This means that $c_{ii}^{m} \neq 0$ for some i, m or $c_{ij}^{m} \neq 0$ for some i, j, m.

Example 1. Let L be a vector space over a field F with the basis $\{x_1, x_2, x_3, x_4\}$. Define an operation $[,]$ by the following rule:

$[x_1, x_1] = x_2,\ [x_1, x_2] = -x_2 - x_3,\ [x_1, x_3] = x_2 + x_3, [x_1, x_4] = 0, [x_2, x_1] = 0, [x_3, x_1] = 0, [x_4, x_1] = x_2 + x_3,$
$[x_j, x_k] = 0$,for all $j, k \in \{2,3,4\}$.

There is an element x in L such that $0 \neq [x, x] = y \in Leib(L)$. Then

$$y = [x, x] = \left[\sum_{i=1}^{4} \alpha_i x_i, \sum_{i=1}^{4} \alpha_i x_i\right]$$
$$= (\alpha_1^2 - \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_4\alpha_1)x_2 + (-\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_4\alpha_1)x_3.$$

Therefore, $Leib(L) = Fx_2 + Fx_3$, $\dim\big(Leib(L)\big) = 2$.

Let $a = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4$, $b = \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 + \gamma_4 x_4$ and $y_1 = [a, a]$, $y_2 = [b, b] \in Leib(L)$. Then we obtain
$[y_1, y_1] = [y_2, y_2] = 0$ and $[y_1, y_2] = \big[[a, a], [b, b]\big] = 0$.
It follows that the structure constants of $Leib(L)$ are zero.

Corollary 1. Let L be a Leibniz algebra with the basis $\{x_1, x_2, \dots, x_n\}$ over a field F. All structure constants of the Leibniz kernel of L are zero.

Proof. By (2), the Leibniz kernel $Leib(L)$ is an abelian Leibniz algebra. Hence for all $y_1, y_2 \in Leib(L)$, we have

$$[y_1, y_2] = 0 = 0.x_1 + 0.x_2 + \cdots + 0.x_n.$$

As a result, the structure constants of $Leib(L)$ are $0, 0, \dots, 0$. As required.

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