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Rational Inequality in Multiplicative Metric Spaces

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Abstract: In this paper, we prove a fixed point theorem for a map that satisfy a rational inequality in multiplicative metric spaces and this generalize various results present in literature.

Keywords: Multiplicative metric spaces, rational inequality, fixed point.

Mathematics Subject Classification: 47H10, 54H25.

1. INTRODUCTION AND PRELIMINARIES

It is well know that the set of positive real numbers $\mathbb{R}_+$ is not complete according to the usual metric. To overcome this problem, in 2008, Bashirov et al. [2] introduced the concept of multiplicative metric spaces as follows:

Definition1.1. ([2]) Let X be a non-empty set. A multiplicative metric is a mapping d: X×X → $\mathbb{R}^+$ satisfying the following conditions:
(i) d(x, y) ≥ 1 for all x, y ∈ X and d(x, y) = 1 if and only if x=y;
(ii) d(x, y) = d(y, x) for all x, y ∈ X;
(iii) d(x, y) ≤ d(x, z). d(z, y) for all x, y, z ∈ X (multiplicative triangle inequality).

Then mapping d together with X i.e., (X, d) is a multiplicative metric space.

Example1.2.([8]) Let R^n_+ be the collection of all n-tuples of positive real numbers.
Let d^*: $\mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ be defind as follows:

$$d^*(x, y) = \left(\left|\frac{x_1}{y_1}\right|^* . \left|\frac{x_2}{y_2}\right|^* \ldots \left|\frac{x_n}{y_n}\right|^*\right),$$

where $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n) \in \mathbb{R}^n_+$ and $|\,.\,| : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$|a|^* = \begin{cases} a & \text{if } a \geq 1; \\ \frac{1}{a} & \text{if } a < 1. \end{cases}$$

Then it is obvious that all conditions of multiplicative metric are satisfied.

Example1.3. ([10]) Let d: $\mathbb{R} \times \mathbb{R} \to [1, \infty)$ be defined as
$d(x, y) = a^{|x-y|}$,where x, y ∈ $\mathbb{R}$ and a > 1. Then d(x, y) is a multiplicative metric and (X, d) is called a multiplicative metric space. We call it usual multiplicative metric spaces.

Example1.4.([10]) Let (X, d) be a metric space .Define a mapping d_a on X by $d_a(x, y) = a^{d(x,y)}$ where a > 1 is a real number and $d_a(x, y) = a^{d(x,y)} = \begin{cases} 1 & \text{if } x = y \\ a & \text{if } x \neq y. \end{cases}$
The metric $d_a(x, y)$ is called discrete multiplicative metric and X together with metric d_a i.e., (X, d_a) is known as a discrete multiplicative metric space.

Example 1.5.([1]) Let $X = C^*[a, b]$ be the collection of all real-valued multiplicative continuous functions over$[a, b] \subseteq R^+$. Then (X, d) is a multiplicative metric space with metric d defined by
$d(f,g) = \sup_{x\in[a,b]} \left|\frac{f(x)}{g(x)}\right|$ for f ,g ∈ X.

Remark1.6. We note that the example 1.1 is valid for positive real numbers and example 1.2 is valid for all real numbers.

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Remark 1.7.([10]) Neither every metric is multiplicative metric nor every multiplicative metric is metric. The mapping d^* defined above is multiplicative metric but not metric as it doesn't satisfy triangular inequality. Consider $d^*(\frac{1}{3}, \frac{1}{2}) + d^*(\frac{1}{2}, 3) = \frac{3}{2} + 6 = 7.5 < 9 = d^*(\frac{1}{3}, 3)$.
On the other, hand the usual metric on R is not multiplicative metric as it doesn't satisfy multiplicative triangular inequality, since $d(2, 3) \cdot d(3, 6) = 3 < 4 = d(2, 6)$.
One can refer to ([8]) for detailed multiplicative metric topology.

Definition1.8.([8]) Let (X, d) be a multiplicative metric space. A sequence $\{x_n\}$ in X said to be a
(i) multiplicative convergent sequence to x, if for every multiplicative open ball $B_\epsilon(x) = \{ y \mid d(x, y) < \epsilon\}$, $\epsilon > 1$, there exists a natural number N such that $x_n \in B_\epsilon(x)$ for all $n \geq N$, i. e, $d(x_n, x) \to 1$ as $n \to \infty$.
(ii) multiplicative Cauchy sequence if for all $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all m, n > N i. e , $d(x_n, x_m) \to 1$ as $n \to \infty$.
A multiplicative metric space is called complete if every multiplicative Cauchy sequence in X is multiplicative converging to $x \in X$.

Remark1.9. We note that the set of positive real numbers $\mathbb{R}_+$ is not complete according to the usual metric. Let $X = \mathbb{R}_+$. Consider the sequence $x_n = \{\frac{1}{n}\}$. It is obvious $\{x_n\}$ is a Cauchy sequence in X with respect to usual metric spaces X and it is not complete metric space as every Cauchy sequence in X does not converge in $\mathbb{R}_+$ i.e., $0 \notin \mathbb{R}_+$. In case of multiplicative metric spaces, consider the sequence $x_n = \{ a^{1/n} \}$,where a >1, it is complete in multiplicative metric spaces, since for $n \geq m$,

$$d^*(x_n, x_m) = \left|\frac{x_n}{x_m}\right|^* = \left|\frac{a^{1/n}}{a^{1/m}}\right|^* = \left|a^{\frac{1}{n}-\frac{1}{m}}\right|^* = a^{\frac{1}{m}-\frac{1}{n}} < a^{\frac{1}{m}} < \epsilon \quad \text{if } m > \frac{\log a}{\log \epsilon},$$

where $|a|^* = \begin{cases} a & \text{if } a \geq 1; \\ \frac{1}{a} & \text{if } a < 1. \end{cases}$

This implies $\{x_n\}$ is a Cauchy sequence in X and it converges to $1 \in \mathbb{R}_+$ as $n \to \infty$. Hence (X, d) is a complete multiplicative metric space.

In 2012, Özavşar and Çevikel[8] introduced the concepts of Banach-contraction, Kannan-contraction, and Chatterjea-contraction mappings in the sense of multiplicative metric spaces as follows:

(Banach-contraction). Let (X, d) be a complete multiplicative metric space and let f: X → X be a multiplicative contraction if there exists a real constant $\lambda \in [0, 1)$ such that

$d(f(x), f(y)) \leq d(x, y)^\lambda$ for all x, y ∈ X. Then f has a unique fixed point.

(Kannan-contraction). Let (X, d) be a complete multiplicative metric space. Suppose the mapping f : X → X satisfies the contraction condition

$d(fx, fy) \leq (d(fx, x) \cdot d(fy, y))^\lambda$, for all x, y ∈ X, where $\lambda \in [0, \frac{1}{2})$.

Then f has a unique fixed point in X and for any x ∈ X, iterative sequence $(f_n(x))$ converges to the fixed point.

(Chatterjea-contraction). Let (X, d) be a complete multiplicative metric space. Suppose the mapping f : X → X satisfies the contraction condition

$d(fx, fy) \leq (d(fy, x) \cdot d(fx, y))^\lambda$, for all x, y ∈ X, where $\lambda \in [0, \frac{1}{2})$.

Then f has a unique fixed point in X and for any x ∈ X, iterative sequence $(f_n(x))$ converges to the fixed point.

2. MAIN RESULTS

Now we prove a fixed point theorem for a map that satisfy rational inequality.

Theorem 2.1. Let f be a continuous self- mapping defined on a complete multiplicative metric space X and f satisfies the following conditions :

$$(2.1)\ d(fx, fy) \leq [d(x,fx)\,.\,d(y,fy)]^{a_1}.\,[d(x,fy).d(y,fx)]^{a_2}.\,[d(x,\ y)]^{a_3}.\ [\tfrac{d(x,fx)\ d(y,Ty)}{d(x,y)}]^{a_4}$$
$$.\{ \max \{d(x,fx)\,,d(y,fy)\,,d(x,fy)\,,d(y,fx)\,,\tfrac{d(x,fx).d(y,fy).d\ (y,fx)}{d(x,y)}\}\}^{a_5}$$

for all x, y ∈ X and $2a_1 + 2a_2 + a_3 + a_4 + a_5 < 1$ where $a_1, a_2, a_3, a_4, a_5 \in [0,1]$.
Then T has unique fixed point.
Proof. Let $\{x_n\}$ be a sequence in X, defined as follows:
Let $x_0 \in X$, $f(x_0) = x_1, f(x_1) = x_2, \cdots, f(x_n) = x_{n+1}$.

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If $x_n = x_{n+1}$ for some n∈ N then x_n is a fixed point of f.
Taking $x_n \neq x_{n+1}$ for all n ∈ N
From (2.1), we have
d(x_{n+1}, x_n) = d(Tx_n,Tx_{n-1})
$\leq [d(x_n, fx_n).d(x_{n-1}, fx_{n-1})]^{a_1}.[d(x_n, fx_{n-1}).d(x_{n-1}, fx_n)]^{a_2}.[d(x_n, x_{n-1})]^{a_3}.[\frac{d(x_n,fx_n)\, d(x_{n-1},Tx_{n-1})}{d(x_n,x_{n-1})}]^{a_4}.$
$\{max\{d(x_n, fx_n), d(x_{n-1}, fx_{n-1}), d(x_n, fx_{n-1}), d(x_{n-1}, fx_n), \frac{d(x_n,fx_n).d(x_{n-1},fx_{n-1}).d(x_{n-1},fx_n)}{d(x_n,x_{n-1})}\}\}^{a_5}$
$\leq [d(x_n, x_{n+1}).d(x_{n-1}, x_n)]^{a_1}.[d(x_n, x_n).d(x_{n-1}, x_n)]^{a_2}.[d(x_n, x_{n-1})]^{a_3}.[\frac{d(x_n,x_{n+1})\, d(x_{n-1},x_n)}{d(x_n,x_{n-1})}]^{a_4}.$
$\{max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_n), d(x_{n-1}, x_{n+1}), \frac{d(x_n,x_{n+1}).d(x_{n-1},x_n).d(x_{n-1},x_{n+1})}{d(x_n,x_{n-1})}\}\}^{a_5}$
$\leq [d(x_n, x_{n+1}).d(x_{n-1}, x_n)]^{a_1}.[d(x_{n+1}, x_n).d(x_{n-1}, x_n)]^{a_2}.[d(x_n, x_{n-1})]^{a_3}.$
$[d(x_n, x_{n+1})]^{a_4}.[d(x_n, x_{n+1})^2.d(x_{n-1}, x_n)]^{a_5}$
d(x_{n+1}, x_n) $\leq [d(x_n, x_{n+1})]^{a_1+a_4+a_2+2a_5}.[d(x_n, x_{n-1})]^{a_1+a_2+a_5+a_3}$,
d(x_n, x_{n+1}) $\leq [d(x_{n-1}, x_n)]^h$,
where h = $\frac{a_1+a_2+a_5+a_3}{1-(a_1+a_4+a_2+2a_5)} < 1$.
Similarly, d(x_{n-1}, x_n) $\leq [d(x_{n-2}, x_{n-1})]^h$,
d(x_n, x_{n+1}) $\leq [d(x_{n-2}, x_{n-1})]^{h^2}$.
Continue like this we get,
d(x_n, x_{n+1}) $\leq [d(x_0, x_1)]^{h^n}$
For n > m, d(x_n, x_m) $\leq$ d(x_n, x_{n-1}) · d(x_{n-1}, x_{n-2}) · · · d(x_m, x_{m+1})
$\leq d(x_0, x_1)^{h^{n-1}+h^{n-2}+\cdots h^m}$
$\leq d(x_0, x_1)^{\frac{h^m}{1-h}}$. This implies d(x_n, x_m) →1(n, m → ∞).

Hence (x_n) is a Cauchy sequence. By the multiplicative completeness of X, there is z ∈ X such that x_n → z (n →∞).
Now we show that z is fixed point of f.
Since f is continuous and x_n → z (n →∞) so, $lim_{n\to\infty} f\, x_n$= fz = $lim_{n\to\infty} x_{n+1}$= z,
i.e., z is a fixed point of f.

Uniqueness: Suppose z, w (z ≠ w) be two fixed point of f, then from (2.1), we have

d(v, w) = d(fv, fw)
$\leq [d(v, fv).d(w, fw)]^{a_1}.[d(v, fw).d(w, fv)]^{a_2}.[d(v, w)]^{a_3}.[\frac{d(v,fv)\, d(w,Tw)}{d(v,w)}]^{a_4}$
$.\{max\{d(v, fv), d(w, fw), d(v, fw), d(w, fv), \frac{d(v,fv).d(w,fw).d(w,fv)}{d(v,w)}\}\}^{a_5}$
d(v, w) $\leq [d(v, w)]^{a_3+2a_2+a_5-a_4}$ this implies that d(v, w) = 1 i.e., v = w.
Hence f has a unique fixed point .

Corrollary1. On Putting $a_2 = a_3 = a_4 = a_5 = 0$ in (2.1), get Kannan-contraction[8] in the sense of multiplicative metric spaces.
Let (X, d) be a complete multiplicative metric space. Suppose the mapping f : X → X satisfies the contraction condition
d(fx,fy) $\leq (d(fx, x) \cdot d(fy, y))^{a_1}$, for all x, y ∈ X, where $a_1 \in [0, \frac{1}{2})$.
Then f has a unique fixed point in X.

Corollary 2. On Putting $a_2 = a_4 = a_5 = 0$ in (2.1), we get Fisher-contraction [4] in the sense of multiplicative metric spaces as follows:
Let f be a continuous self- mapping defined on a complete multiplicative metric space X, further f satisfies the following conditions
d(fx, fy) $\leq [d(x, fx).d(y, fy)]^{a_1}.[d(x, y)]^{a_3}$,for all x, y ∈ X and $2a_1 + a_3 < 1$, where $a_1, a_3 \in [0,1]$.
Then T has unique fixed point.

Corollary 3. On Putting $a_2 = a_3 = a_4 = a_5 = 0$ in (2.1), we get Chatterjea-contraction[8] in the sense of multiplicative metric spaces.
Let (X, d) be a complete multiplicative metric space. Suppose the mapping f : X → X satisfies the contraction condition
d(fx,fy) $\leq (d(fy, x) \cdot d(fx, y))^{a_1}$, for all x, y ∈ X, where $a_1 \in [0, \frac{1}{2})$.

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Then f has a unique fixed point in X.

Corollary 4. On Putting $a_1 = a_2 = a_4 = a_5 = 0$ in (2.1), we get Banach-contraction[8] in the sense of multiplicative metric spaces as follows:
Let (X, d) be a complete multiplicative metric space and let f: X → X be a multiplicative contraction if there exists a real constant $a_3 \in [0, 1)$ such that

d(f(x), f(y)) ≤ $d(x,y)^{a_3}$ for all x, y ∈ X. Then f has a unique fixed point.

Corollary 5. On Putting $a_4 = a_5 = 0$, in (2.1), we get Ciric-contraction[3] in the sense of multiplicative metric spaces as follows:
Let f be a continuous self- mapping defined on a complete multiplicative metric space X, further f satisfies the following conditions
d(fx, fy) ≤ $[d(x,fx)\,.d(y,fy)]^{a_1}.[d(x,fy).d(y,fx)]^{a_2}.[d(x,\ y)]^{a_3}$,
for all x, y ∈ X and $2a_1 + 2a_2 + a_3 < 1$ where $a_1, a_2, a_3 \in [0,1]$.
Then T has unique fixed point.

Corollary 6. On Putting $a_1 = a_4 = a_5 = 0$ in (2.1), we get Reich-contraction[9] in the sense of multiplicative metric spaces as follows:
Let f be a continuous self- mapping defined on a complete multiplicative metric space X, further f satisfies the following conditions
d(fx, fy) ≤ $[d(x,fy).d(y,fx)]^{a_2}.[d(x,\ y)]^{a_3}$, for all x, y ∈ X and $2a_2 + a_3 < 1$ where $a_2, a_3 \in [0,1]$. Then T has unique fixed point.

Corollary 7. On Putting $a_1 = a_2 = a_5 = 0$ in (2.1), we get jaggi-contraction[6] in the sense of multiplicative metric spaces as follows:
Let f be a continuous self- mapping defined on a complete multiplicative metric space X, further f satisfies the following conditions
d(fx, fy) ≤ $[d(x,\ y)]^{a_3}.\ [\frac{d(x,fx)\ d(y,Ty)}{d(x,y)}]^{a_4}$,
for all x, y ∈ X and $a_3 + a_4 < 1$ where $a_3, a_4 \in [0,1]$.
Then T has unique fixed point.

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