IARJSET

ISSN (Online) 2393-8021
ISSN (Print) 2394-1588

International Advanced Research Journal in Science, Engineering and Technology
Vol. 3, Issue 6, June 2016

Fixed Points of Automorphisms Permuting the Generators Cyclically in Free solvable Lie algebras

Zerrin Esmerligil

Department of Mathematics, Çukurova University, Adana, Turkey

Abstract: We investigate fixed points of an automorphism of a free solvable Lie algebra which permutes the generators cyclically. Let Θ be a cyclic permutation of order n which belongs to the nth symmetric group S_n. We give form of the fixed points of an automorphism of a free solvable Lie algebra which is induced by the permutation Θ.

Keywords: Free Solvable Lie algebra, automorphism, fixed point, cyclic permutation.

I.INTRODUCTION

Let F be the free Lie algebra freely generated by a set $X = \{x_1, \dots, x_n\}, n \geq 2$, over a field K. The derived series of F is defined as the following:

$\delta^0(F) = F, \delta^1(F) = F' = [F, F]$ and for $m > 1$ we define
$$\delta^m(F) = [\delta^{m-1}(F), \delta^{m-1}(F)].$$

Fixed points subalgebras of free Lie algebras are studied by Bryant[1] and Drensky [3]. In [2] Bryant and Papistas have obtained some results about fixed point subalgebras of relatively free Lie algebras. Later, Ekici and Sönmez [4] have given a criterion detecting nontrivial fixed points of IA- automorphisms of a free metabelian Lie algebra.

Fixed point subalgebras of automorphisms preserving the length of words of free solvable groups are described by Tomaszewski [5]. In this work we obtained corresponding results for free solvable Lie algebras.
By L_m we denote the free solvable Lie algebra $F/\delta^m(F)$ of rank n and solvability class m.

Let θ be an automorphism of F of order k induced by a permutation $\sigma \in S_n$, where S_n is the nth symmetric group. The automorphism θ induces an automorphism $\bar{\theta}: L_m \to L_m$ which is defined as $\bar{\theta}(\bar{\omega}) = \theta(\omega) + \delta^m(F)$, where $\omega \in F$, $\bar{\omega} = \omega + \delta^m(F)$. For an element $\bar{\omega}$ of L_m if $\bar{\theta}(\bar{\omega}) = \bar{\omega}$ then $\bar{\omega}$ is called a fixed point of L.
It can be easily seen that if θ has order k then every element of the form

$$\bar{\omega} + \bar{\theta}(\bar{\omega}) + \bar{\theta}^2(\bar{\omega}) + \dots + \bar{\theta}^{k-1} \quad (1)$$

is a fixed point for $\bar{\theta}$, where $k \geq 2$, $\bar{\omega} \in F/\delta^m(F)$.

It is not obvious that only such elements are the fixed points. In this work we prove that every fixed point of $\bar{\theta}$ has the form (1).

II. MAIN RESULT

Assume that $\sigma \in S_n$ is a product of disjoint cycles σ_i of length r_i, $i = 1, \dots, s$. Let $G = F/F'$.

Lemma
Assume that θ is an automorphism of F which is induced by σ. If $\hat{\theta}$ is an automorphism of G, induced by θ then every fixed points of $\hat{\theta}$ has the form

$$\sum_{i=1}^{s} \left(\hat{\omega}_i + \hat{\theta}(\hat{\omega}_i) + \hat{\theta}^2(\hat{\omega}_i) + \dots + \hat{\theta}^{r_i - 1}(\hat{\omega}_i) \right),$$
where $\hat{\omega}_i = \beta_i \hat{x}_{i_1}, \beta_i \in K, i = 1, \dots, s$.

Proof
Let $\hat{\theta}$ be an automorphism of G, induced by θ. If $\hat{v} \in G$ is a fixed point of $\hat{\theta}$ then $\hat{\theta}(\hat{v}) = \hat{v}$. The element $\hat{v}$ can be uniquely written as

$$\hat{v} = \sum_{j=1}^{n} c_j \hat{x}_j, \qquad c_j \in K$$

By taking into account the cycles of σ we arrange the generators which we see in $\hat{v}$ as $\hat{v} = \sum_{i=1}^{s} \sum_{t=1}^{r_i} c_{i_t} \hat{x}_{i_t}$.
Using the equality $\hat{\theta}(\hat{v}) = \hat{v}$ we get

$$c_{i_t} = c_{i_l} = \beta_i, \quad 1 \leq t, l \leq r_i, \quad i = 1, \dots, s.$$

Therefore $\hat{v} = \sum_{i=1}^{s} \beta_i \left(\sum_{t=1}^{r_i} \hat{x}_{i_t} \right)$. Since $\hat{\theta}(\hat{x}_{i_1}) = x_{i_2}, \dots, \hat{\theta}\left(\hat{x}_{i_{r_i}}\right) = \hat{x}_{i_1}$ then

$$\sum_{t=1}^{r_i} \hat{x}_{i_t} = \left(I + \hat{\theta} + \hat{\theta}^2 + \dots + \hat{\theta}^{r_i - 1}\right)\left(\hat{x}_{i_1}\right)$$

and so $\hat{v} = \sum_{i=1}^{s} \left(I + \hat{\theta} + \hat{\theta}^2 + \dots + \hat{\theta}^{r_i - 1}\right) \left(\beta_i \hat{x}_{i_1}\right)$.■

By the above Lemma it is clear that if σ is a cycle of order n and $\hat{\theta}$ is an automorphism of G induced by θ then every fixed point of $\hat{\theta}$ has the form

$$\hat{\omega} + \hat{\theta}(\hat{\omega}) + \hat{\theta}^2(\hat{\omega}) + \dots + \hat{\theta}^{n-1}(\hat{\omega}),$$
where $\hat{\omega} = \beta \hat{x}_k, \quad \beta_i \in K, \quad x_k \in X.$

Theorem

IARJSET

ISSN (Online) 2393-8021
ISSN (Print) 2394-1588

International Advanced Research Journal in Science, Engineering and Technology
Vol. 3, Issue 6, June 2016

Assume that $\sigma \in S_n$ be a cycle of order n and θ be an automorphism of F induced by σ.

If $\bar{\theta}$ is an automorphism of L_m induced by θ then every fixed point of $\bar{\theta}$ has the form

$$\bar{\omega} + \bar{\theta}(\bar{\omega}) + \bar{\theta}^2(\bar{\omega}) + \ldots + \bar{\theta}^{n-1}(\bar{\omega}),$$

where $\bar{\omega} = \alpha\bar{x}_k + h, \ \alpha \in K, \ x_k \in X, \ h \in L'_m$.

Proof

Let $\bar{v} \in L_m$ be a fixed point of $\bar{\theta}$. We use induction on m. For $m = 1$ $L_1 = F/_{F'}$ is a free abelian Lie algebra. So by the Lemma the result is clear.

Suppose that the assertion is true for all positive integers less than m. Let $\tilde{\theta}$ be an automorphism of L_{m-1} induced by θ.

By induction hypothesis the fixed points of the automorphism $\tilde{\theta}$ of the algebra $L_{m-1} = F/_{\delta^{m-1}F}$ are the elements of the form

$$\tilde{\omega}_1 + \tilde{\theta}(\tilde{\omega}_1) + \tilde{\theta}^2(\tilde{\omega}_1) + \cdots + \tilde{\theta}^{n-1}(\tilde{\omega}_1),$$

where $\tilde{\omega}_1 = \alpha\tilde{x}_k + h_1, \ \alpha \in K, \ x_k \in X, \ h_1 \in L'_{m-1}$.

Let $\tilde{u}$ be a fixed point of $\tilde{\theta}$ in L_{m-1}. Assume that $\tilde{u} = \tilde{\psi}(\tilde{\omega}_1)$, where

$\tilde{\psi} = I + \tilde{\theta} + \tilde{\theta}^2 + \cdots + \tilde{\theta}^{n-1}$.

Since

$$L_{n,m-1} = F/_{\delta^{m-1}F} \cong (F/_{\delta^m F})/(\delta^{m-1}F/_{\delta^m F}),$$

then the preimage of $\tilde{u}$ in $F/_{\delta^m F}$ is of the form

$$a = \psi(\omega_1) + g + \delta^m F,$$

where $g \in \delta^{m-1}F/_{\delta^m F}$. Then we have $\bar{\theta}(\bar{g}) = \bar{g}$ in the algebra $\delta^{m-1}F/_{\delta^m F}$. By the Lemma the element $\bar{g}$ has the form

$$\bar{g} = \bar{\omega}_2 + \bar{\theta}(\bar{\omega}_2) + \bar{\theta}^2(\bar{\omega}_2) + \cdots + \bar{\theta}^{n-1}(\bar{\omega}_2), \quad (2)$$

where $\bar{\omega}_2 = \beta\bar{b}, \ \beta \in K, \ \bar{b} \in \delta^{m-1}F/_{\delta^m F}$. Hence

$$a = \psi(\omega_1 + \omega_2) + \delta^m F \quad (3)$$

Now let $\bar{v}$ be a fixed point of $\bar{\theta}$ in L_m. The element $\bar{v}$ can be written as $\bar{v} = \bar{v}_1 + \bar{v}_2$,

where $v_1 \in F(\mathrm{mod}\,\delta^{m-1}F), \ v_2 \in \delta^{m-1}F$. Since $\bar{\theta}(\bar{v}) = \bar{v}$ we get $\tilde{\theta}(\tilde{v}_1) = \tilde{v}_1$ and

$\bar{\theta}(\bar{v}_2) = \bar{v}_2$. By (2) and (3) we see that $\bar{v}_1 + \bar{v}_2$ has the form

$$\bar{v}_1 + \bar{v}_2 = \bar{\psi}(\bar{\omega}_1 + \bar{\omega}_2), \text{ where } \bar{\omega}_1 = \alpha\bar{x}_k + h_1, \ \bar{\omega}_2 = \beta\bar{b}, \ \alpha, \beta \in K, h_1 \in L'_m, \bar{a} \in \delta^{m-1}F/_{\delta^m F}.$$

It can be easily seen that every element of the form

$$\bar{\omega} + \bar{\theta}(\bar{\omega}) + \bar{\theta}^2(\bar{\omega}) + \ldots + \bar{\theta}^{n-1}(\bar{\omega})$$

is a fixed point of $\bar{\theta}$.

REFERENCES

[1] R.M. Bryant, On the fixed points of finite group acting on a free Lie algebra. J. London Math Soc., 43(2): 215-224, 1991.

[2] R.M. Bryant, A.I. Papistas, On the fixed points of a finite group acting on a relatively free Lie algebra, Glasg Math J.,42:167-181, 2000.

[3] V. Drensky, Fixed algebras of residually nilpotent Lie algebras. Proc Amer Math Soc.,120: 1021-1028, 1994.

[4] N.Ekici, D.Sönmez, Fixed points of IA- endomorphisms of a free metabelian Lie algebra. Proc Indian Acad Sci (Math Sci.),121(4):405-416, 2011.

[5] T. Witold, Fixed points of automorphisms preserving the length of words in free solvable groups Arch. Math., 99:425-432, 2012.