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On Central Automorphısms Of Free Center-By-Metabelıan Lıe Algebras

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Abstract: We study central automorphisms of free center-by metabelian Lie algebras. Our main result exhibits the form of such automorphisms.

Keywords: Central Automorphism, Center-by-metabelian, Lie algebra

I.INTRODUCTİON

Let F be the free Lie algebra with two free generators x an y over a field K, and let L be the free center-by-metabelian Lie algebra $F/_{[F'',F]}$. Clearly L is freely generated by the set $\{\bar{x},\bar{y}\}$, where $\bar{x}=x+[F'',F]$, $\bar{y}=y+[F'',F]$. We write x,y instead of $\bar{x},\bar{y}$. By $Aut(L)$ we denote the automorphism group of all automorphisms of L.

Definition

Let $\theta\in Aut(L)$. If θ induces the identitiy mapping on the algebra $L/_{Z(L)}$ then it is called a central automorphism of L, where $Z(L)$ is the center of L. If θ is a central automorphism of L then for every $u\in L$ we have $\theta(u)-u\in Z(L)$. It can be easily seen that $Z(L)=F''/_{[F'',F]}$. Hence form of any central automorphism of L is

$$\theta(x)=x+u,\ \ \theta(y)=y+v,\qquad u,v\in L''$$

For any $g,h\in L$ we write

$$[g,h^n]=[g,\underbrace{h,\dots h}_{n-times}].$$

A basis in L'' is formed by the elements

$$\left[[[g,y],x^{n_1},y^{n_2}],[x,y]\right],\quad (n_1,n_2\geq 0),\qquad (1)$$

where $g\in L$ and the sum n_1+n_2 is odd. (See [1], [7] and [8] for details).

Although there are many publications about central automorphisms of groups [2,3,4,5,6] the corresponding problems for relatively free Lie algebras are very rare. In [9], Ekici and Öztekin have given some characterizations of central automorphisms of free nilpotent Lie algebras.

In this work we prove that the following results.

Proposition

In the center-by-metabelian Lie algebra L every map $\varphi: L\to L$ defined by

$$\varphi: x\to x+\sum \alpha_g\left[[[g,y]x^{m_1},y^{m_2}][x,y]\right]\quad \alpha_g\in K, g\in L$$

$$x\to x+\sum \beta_h\left[[[h,y]x^{t_1},y^{t_2}][x,y]\right]\quad \beta_h\in K, h\in L$$

is an automorphism.

Proof

It can be easily seen that the Jacobian matrix of φ is invertible over $U(L/_{L'})$. Hence φ is an automorphism.

Theorem

Any central automorphism θ of L has the form

$$\theta: x\to x+\sum_{n_1,n_2}\alpha_g\left[g,\left[\left[[[x,y],y^{n_2},]x^{n_1}\right],y\right]\right],\qquad (2)$$
$$y\to y+\sum_{n_1,n_2}\beta_h\left[g,\left[\left[[[x,y],y^{n_2},]x^{n_1}\right],y\right]\right]$$

where, $g,h\in L'$, α_g, $\beta_h\in K$, n_1+n_2, r_1+r_2 are odd.

Proof

Let θ be a central automorphism of L. Then it has the form

$$\theta: x\to x+u,$$
$$y\to y+v,$$

Where $u,v\in L''$. Then the Lie algebra L'' has a linear basis of the form (1). Thus, the elements u,v can be written as linear combinations of elements of the form (1). Therefore we define θ as

$$\theta: x\to x+\sum_{n_1,n_2}c_g\left[\left[[[g,y],x^{n_1}],y^{n_2}\right],[x,y]\right],$$
$$y\to y+\sum_{r_1,r_2}d_h\left[\left[[[h,y],x^{r_1}],y^{r_2}\right],[x,y]\right],$$

where $g,h\in L/_{L''}$, $c_g,d_h\in K$.

Now let us apply the Jacobi identity to the elements

$$\left[\left[[[g,y],x^{n_1}],y^{n_2}\right],[x,y]\right] \text{ and } \left[\left[[[h,y],x^{r_1}],y^{r_2}\right],[x,y]\right]$$

consecutively we obtain

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$$u = \left[\left[\left[[g,y],x^{n_1}\right],y^{n_2}\right],[x,y]\right] = \left[g,\left[\left[\left[[x,y],y^{n_2}\right],x^{n_1}\right],y\right]\right]$$

And

$$v = \left[\left[\left[[h,y],x^{r_1}\right],y^{r_2}\right],[x,y]\right] = \left[h,\left[\left[\left[[x,y],y^{r_2}\right],x^{r_1}\right],y\right]\right].$$

Since $u,v \in L''$ we see that the elements g and h have to belong L'. Therefore θ has the form

$$\theta: x \to x + \sum_{n_1,n_2\geq 0} \alpha_g \left[g,\left[\left[\left[[x,y],y^{n_2}\right],x^{n_1}\right],y\right]\right],$$
$$y \to y + \sum_{r_1,r_2\geq 0} \beta_h \left[h,\left[\left[\left[[x,y],y^{r_2}\right],x^{r_1}\right],y\right]\right],$$

where $g,h \in L'$, $\alpha_g,\beta_h \in K$.

Lemma
Let $\theta \in Aut(L)$. If $[\theta(\omega),\omega] = 0$ for all $w \in L$, then it is central.

Proof
Let $\theta \in Aut(L)$ such that $[\theta(\omega),\omega] = 0$ for all $\omega \in L$. We define θ as

$$\theta: x \to \alpha x + \beta y + u,$$
$$y \to \gamma x + \delta y + v,$$

where $u,v \in L'$, $\alpha,\beta,\gamma,\delta \in K$. By the assumption

$$[\theta(x),x] = \beta[y,x] + [u,x] = 0,$$
$$[\theta(y),y] = \gamma[x,y] + [v,y] = 0,$$

These equalities lead $\beta = \gamma = 0$. Hence θ has the form

$$\theta: x \to \alpha x + u,$$
$$y \to \delta y + v.$$

From the equalitiy $[\theta(x+y),x+y] = 0$ we get

$$0 = [\alpha x + u + \delta y + v, x + y]$$
$$= \alpha[x,y] + [u,x] + [u,y] + \delta[y,x] + [v,x] + [v,y]$$
$$= (\alpha - \delta)[x,y] + [u,x+y] + [v,x+y].$$

Hence $(\alpha - \delta) = 0$. Thus θ has the form

$$\theta: x \to \alpha x + u,$$
$$y \to \alpha y + v,$$

where $\alpha \neq 0$.

Since $\theta \in Aut(L)$ then

$$[\theta(x),\theta(y)] \equiv \alpha^2[x,y](mod[F'',F]). \qquad (3)$$

Let us calculate $[\theta(x),\theta(y)]$.

$$[\theta(x),\theta(y)] = [\alpha x + u, \alpha y + v]$$
$$= \alpha^2[x,y] + \alpha[x,v] + \alpha[u,y] + [u,v]$$

By (3) we get $\alpha([x,v] + [u,y]) + [u,v] \in [F'',F]$.

Hence $u,v \in F''$.
Now consider the element $\omega = x - [x,y]$.

$$0 = [\theta(\omega),\omega]$$
$$= [\theta(x) - [\theta(x),\theta(y)],\omega]$$
$$= [\alpha x + u - [\alpha x + u, \alpha y + v],\omega]$$
$$= \left[\alpha x + u - \alpha^2[x,y], x - [x,y]\right]$$
$$= -\alpha[x,[x,y]] - \alpha^2\left[[x,y],x\right]$$
$$= (\alpha - \alpha^2)\left[[x,y],x\right]$$

Thus $\alpha = 1$. Therefore θ has the form

$$\theta: x \to x + u,$$
$$y \to y + v,$$

where $u,v \in L''$.

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