

Centers in Subdivision and Inserted Graphs

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Abstract: In this paper we study some concepts involving distance in subdivision graphs and inserted graphs and their centers. We prove some results on center, periphery and radius of the subdivision graph $S(G)$ and inserted graph $I(G)$ of a graph G for complete, cycle, complete bipartite, star and wheel graphs. An important theorem has been proved that in a connected graph the centers of G and $S(G)$ have no common vertex. Graphs which are the periphery of some subdivision graph are characterized.

Keywords: Distance, Eccentricity, Radius, Diameter, Center, Subdivision graph and Inserted graph.

I. INTRODUCTION

We consider ordinary graph $G = (V, E)$, we mean a finite, undirected, connected graph without loop or multiple edges with vertex set V_G and an edge set E_G . A graph G with exactly one vertex is called a trivial graph, implying that the order of a nontrivial graph is at least 2. For a graph G and a pair u, v of vertices of G , the distance $d(u, v)$ between u and v is the length of a shortest $u-v$ path in G . An $u-v$ path of length $d(u, v)$ is an $u-v$ geodesic in G . The degree of a vertex u , denoted by $\deg(u)$ is the number of vertices adjacent to u . A vertex of even degree is called an even vertex, while a vertex of odd degree is called an odd vertex. For a vertex v in a graph G , the eccentricity $e(v)$ of v is the distance between v and a vertex farthest from v in G . Thus $e(v) = \max\{d(u, v) \mid u \in V\}$. The minimum eccentricity among the vertices of G is its radius and the maximum eccentricity is its diameter, which are denoted by $\text{rad}(G)$ and $\text{diam}(G)$ respectively. A vertex v in G is a central vertex if $e(v) = \text{rad}(G)$ and a sub graph induced by the central vertices of G is the center and it is denoted by $\text{cen}(G)$. If every vertex of G is a central vertex then $\text{cen}(G) = G$ and G is called self-centered. A vertex v in G is a peripheral vertex if $e(v) = \text{diam}(G)$ and a sub graph induced by the peripheral vertices of G is the periphery and it is denoted by $\text{per}(G)$.

The subdivision graph of G is $S(G) = (V \cup E, E')$ where $E' = \{\{e, v\} : e \in E \text{ and } v \text{ is incident with } e\}$. Each edge of G is replaced by a path of length 2. A vertex u of a graph G is called a universal vertex if u is adjacent to all other vertices of G . A graph can be constructed by inserting a new vertex on each edge of G and the resulting graph is called a box graph of G , denoted by $B(G)$. For an edge e of G , \bar{e} denotes the vertex of $B(G)$ corresponding to the edge e . Let I_G be the set of all inserted vertices in $B(G)$. A graph $I(G)$ with vertex set I_G is called the inserted graph in which any two vertices are adjacent if they are joined by a path of length two in $B(G)$.

Moreover, if

$$V_G = \{v_1, v_2, \dots, v_n\} \text{ and}$$

$$E_G = \{e_1, e_2, \dots, e_m\}$$

then

$$V_{B(G)} = \{v_1, v_2, \dots, v_n, \bar{e}_1, \bar{e}_2, \dots, \bar{e}_m\}.$$

Example 1.1

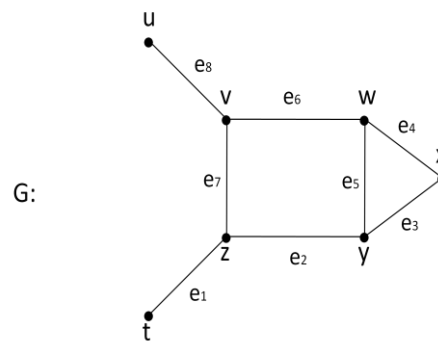


Figure 1 A graph G

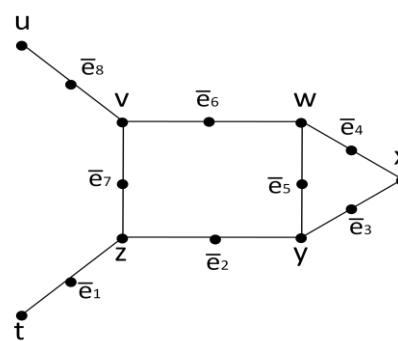


Figure 2 The box graph of a graph G

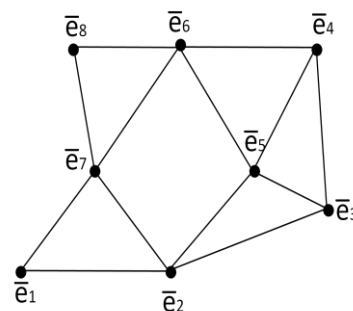


Figure 3 The inserted graph of a graph G

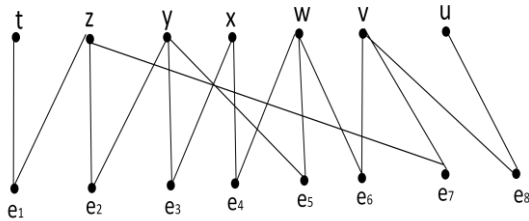


Figure 4 The subdivision graph of a graph G

II. MAIN RESULTS

Theorem 2.1

If G is a complete graph of order $n > 3$, then $C(S(G))=C(G)$ and $per(S(G))=E(G)$.

Proof:

Let G be a complete graph K_n and $V(G)=\{v_1, v_2, \dots, v_n\}$. If G is complete then it is clear that $C(G)=V(G)$. Let S(G) be the subdivision graph of G. Then $V(S(G))=\{v_1, v_2, \dots, v_n, e_1, e_2, \dots, e_m\}$ where $m=nC_2$. Let $v_i, v_j \in G, 1 \leq i, j \leq n$ be two adjacent vertices. Then there exist e such that e lies between v_i and v_j and so $d_G(v_i, v_j) = 1$. Now, by the definition of S(G), $d_{S(G)}(v_i, v_j) = 2$. Hence $d_{S(G)}(v_i, v_j) = 2 d_G(v_i, v_j)$. Also, there exist e_i such that e_i is adjacent to v_j but not adjacent to a vertex v_i in S(G). Therefore $d_{S(G)}(v_j, e_i) = 1$. $d_{S(G)}(v_i, e_i) = d_{S(G)}(v_i, v_j) + d_{S(G)}(v_j, e_i) = 2 + 1 = 3$ and so $e_{S(G)}(v_i) = 3$. Let $e_j \in S(G)$ be adjacent to the vertex v_i in S(G). That is, $d_{S(G)}(v_i, e_j) = 1$. $d_{S(G)}(e_i, e_j) = d_{S(G)}(e_i, v_i) + d_{S(G)}(v_i, e_j) = 4$. Therefore, $e_{S(G)}(e_i) = e_{S(G)}(e_j) = 4$. Clearly, $C(S(G)) = \{v_1, v_2, \dots, v_n\}$. This implies, $C(S(G))=V(G)=C(G)$ and $Per(S(G)) = E(G) = \{e_1, e_2, \dots, e_m\}$.

Note:

If $G \subseteq K_n$, then $C(G) \subseteq C(S(G))$ and $Per(G) \subseteq per(S(G))$.

Example 2.2

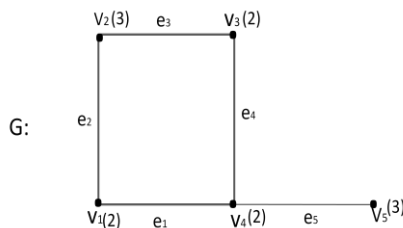


Figure 5 A graph G

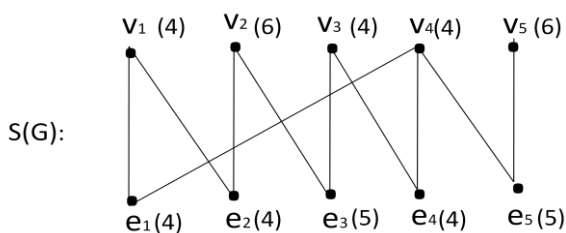


Figure 6 The subdivision graph of a graph G

Theorem 2.3

Let G be a cycle with at least three vertices. If G has a nontrivial center and a radius smaller than its subdivision graph, then $S(C(G))$ is an induced sub graph of $C(S(G))$. Moreover, if G is self-centered then S(G) is also self-centered, and $S(C(G))=C(S(G))$ iff G is self-centered.

Proof:

Let $r(S(G))=R+n$ ($n \geq 2$) where R is a radius of G. then for a vertex \bar{uv} in $S(C(G))$ we have $e_G(u) = e_G(v) = R$, which gives $e_G(uv) = e_{S(G)}(\bar{uv}) \leq R+n = r(S(G))$ and that is why \bar{uv} is in $C(S(G))$. Moreover if G is self-centered then S(G) is an induced sub graph in $C(S(G))$, hence S(G) is self-centered. Further, if G is self-centered, then we have $S(C(G))=S(G)=C(S(G))$. Conversely suppose that, G is not self-centered. Then it contains an edge $x y$ joining a central vertex x to a non-central vertex y, hence $e_G(x) = R$ and $e_G(y) = R+n$. Then $e_G(xy) = e_{S(G)}(\bar{xy}) \leq R+n$ and \bar{xy} is in $S(C(G))$. hence $S(C(G))=C(S(G))$ does not hold.

Theorem 2.4

If G is a cycle C_n , then $C(S(G))=per(S(G))$.

Proof:

Obviously the result is true. Since S(G) will form another cycle. $\therefore C(S(G))=per(S(G))$.

Theorem 2.5

For a complete bipartite graph $G = K_{n,n}$, then $r(S(G))=2r(G)$.

Proof:

Let G be a complete bipartite graph $K_{n,n}$. Then $r(G)=2=diam(G)$. Hence, $C(G)=V(G)$. Now, $V(G) \cup E(G)$ is the vertex set of S(G). let u and v be adjacent vertices of G, Then $d_G(u, v)=1$ and $d_{S(G)}(u, v)=2$. let u and v be non-adjacent vertices of G, Then $d_G(u, v)=2$ and $d_{S(G)}(u, v)=4$. Hence, $d_{S(G)}(u, v)=2 d_G(u, v)$. Let X and Y be the partition of $V(G)$. Let v be the farthest vertex of u in G. Then $u \in X$ implies $v \in Y$ and so $d_G(u, v)=2$ and $d_{S(G)}(u, v)=4$. Hence, $e_G(u) = 2$ and $e_{S(G)}(u) = 4$. Therefore, $r(G)=2$ and $r(S(G))=4$. Hence, $r(S(G))=2r(G)$.

Corollary 2.6

For a complete bipartite graph $G = K_{n,n}$, then $Per(S(G))=C(S(G))$.

Proof:

Let G be a complete bipartite graph $K_{n,n}$. Then $r(G)=2=diam(G)$. Hence, $C(G)=V(G)=Per(G)$. Using the above theorem, $r(S(G))=4=diam(S(G))$. Hence, $C(S(G))=V(G)=Per(S(G))$.

Theorem 2.7

Let G be a connected graph. Let $e = uv$ be the only cut-edge and u, v be the only central vertices of G, then $C(G) \cap C(S(G)) = \emptyset$

Proof:

Let G be a connected graph and $e = uv$ is the only cut-edge of G. Then $G-e$ contains two components G_1 and G_2 each of which contains vertices of $C(G)$. let w be the vertex of G such that $d(w, u)=e(u)$. and let P_1 be a w-u geodesic in

G . at least one of G_1 and G_2 contains no vertices of P_1 , say G_2 contains no vertices of P_1 . Let u be a central vertex of G , that belongs to G_1 , then $e(u)=r(G)$. This implies $u \in C(G)$. Let x be an eccentric vertex of v . $d(x, v)=e(v)$. and P_2 be a $v-x$ geodesic in G . let v be a central vertex of G , that belongs to G_2 , then $e(v)=r(G)$. This implies $v \in C(G)$. Hence, $u, v \in C(G)$. we add an edge e (say e_i) to the central vertices u and v of G . Let $V(G) \cup E(G)$ is the vertex set of $S(G)$. Let e_i be a central vertex of $S(G)$. i.e., $e_i \in C(S(G))$. Hence, $C(G) \cap C(S(G)) = \Phi$

Theorem 2.8

Let W_n , $n \geq 5$ be a wheel graph with the vertex set $\{v_1, v_2, \dots, v_n\}$ and v_n be the universal vertex. Then $C(S(W_n)) = \{v_n\} = C(W_n)$.

Proof:

Let $G = W_n$, $n \geq 5$. let $V(G) = \{v_1, v_2, \dots, v_n\}$. and v_n is the universal vertex. Then $e(v) = 1 = r(G)$ and so $C(G) = \{v_n\}$. Let $\{v_1, v_2, \dots, v_n, e_1, e_2, \dots, e_m\}$ be a vertex set of $S(G)$. For every $v \in V(S(G))$ $e(v_n) = 3$ and $v \in V(S(G))$, $e(v) = 4$ for $v \neq v_n$ and therefore $C(S(G)) = \{v_n\}$. Hence, $C(G) = C(S(G)) = \{v_n\}$.

Theorem 2.9

If G is a star graph $S_{1,n}$, then $I(G)$ is a complete graph K_n .

Proof:

Let G be a star graph and $\{v_1, v_2, \dots, v_n\}$ be the vertex set of G . let v_n be the universal vertex. Therefore, $e(v_n) = 1$ and $e(v_i) = \dots = e(v_{n-1}) = 2$. Let $\{v'_1, v'_2, \dots, v'_m\}$ be the vertex set of $I(G)$. Then by the definition of $I(G)$, we get every vertex of $I(G)$ has eccentricity 1.

Hence distance between any two vertices is 1. That is, any pair of vertices are adjacent. Hence, $I(G)$ is a complete graph with n vertices.

Theorem 2.10

Let G be a star graph and $I(G)$ be a complete graph. then $r(S(G)) = 2.r(I(G))$

Proof:

Let G be a star graph. From theorem 2.9, $I(G)$ is a complete graph. Let $\{v_1, v_2, \dots, v_n\}$ be the vertex set of G . let v_n be the universal vertex. Then $r(I(G)) = r(G) = 1$. Also, $d_{S(G)}(v_n, v_i) = 2 d_G(v_n, v_i)$ or $v_i \in V$. Taking maximum on both sides, we have $e_{S(G)}(v_i) = 2 e_G(v_i)$. Hence, $r(S(G)) = 2.r(G)$ since $r(I(G)) = r(G)$.

Theorem 2.11

A double star graph G has a cut-edge $e = uv$ and u, v be the only central vertices of G , if each component of $G-e$ is a star G_1 and G_2 , then $r(I(G_i)) = r(G_i)$ ($i = 1, 2$).

Proof:

Let G be a double star graph and $e = uv$ be the only cut-edge of G . Then $G-e$ contains two components G_1 and G_2 . Let G_1 and G_2 be star graphs. Let u and v are universal vertices of G_1 and G_2 respectively. Then $r(G_1) = r(G_2) = 1$. From theorem 2.9, $r(I(G_1)) = r(I(G_2)) = 1$. Therefore, hence $r(I(G_i)) = r(G_i)$ ($i = 1, 2$).

III. CONCLUSION

Many researchers are concentrating the distance concept in graphs. In this paper we study the subdivision and inserted graphs for different types of graphs and investigate their properties. Many results have been found and compared for the above said graphs. In similar way the center concepts for other graphs can be studied.

REFERENCES

- [1] M.R.Adhikari and L.K. Pramanik, "The Connectivity of inserted Graphs", J.Chung. Math.Soc. 18(2005), 61-68.
- [2] M.R.Adhikari, L.K. Pramanik and S.Parui, "On box graph and its Square", Rev. Bull. Cal. Math. Soc.13 (2005), 61-64.
- [3] F. Buckley and F. Harary, Distance in Graphs, Addison-Wesley Publishing Company, New York, (1990).
- [4] F.Buckley, Z.Miller, and P.J.Slater, "On graphs containing a given Graph as center", Journal of Graph Theory, 5(4): 427-434, 1981.
- [5] Gary Chartrand and Ping Zhang, Introduction to Graph Theory, Tata McGraw-Hill, New Delhi (2006).
- [6] J.D.Horton and K.Kilakos, "Minimum Edge Dominating Sets", SIAM Journal on Discrete Mathematics / Vol.6/ISSUE 3, August 1993(375-387).
- [7] L.K. Pramanik, "Centers In Inserted Graphs", Faculty of Sciences and Mathematics (2007), 21-30.

BIOGRAPHY

Dr. A. Anto Kinsley has twenty nine years teaching experience in the department of Mathematics, St. Xavier's College, Palayamkottai, Tamil Nadu and he has published twenty two research papers in both national and international journals. He has completed two UGC sponsored minor research projects. His area of specialisation is distance and algorithms in graphs. Ms. J. Joan Princiya is one of his research students. She is doing research in center concepts in graphs with respect to edges. She has published a research paper in an international journal.