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Some Special Conditions in Topological Summed

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Abstract: This study includes free union of a disjoint non-empty collection of topological spaces and research on the disjoint union topology (topological summed). Definitions, theorems and some results for topological summed have been obtained by using the known definitions and theorems for the topological spaces.

Keywords: Free Union, Disjoint Union Topology, Topological Summed, Topology.

1. INTRODUCTION

Let $\{(X_\lambda,\tau_\lambda)\}_{\lambda\in\Lambda}$ be a collection of topological space. Let (X,τ) be a topological space with $X=\bigcup X_\lambda$, $\lambda\in\Lambda$. (X,τ) is a free union of (X_λ,τ_λ) iff the following condition satisfied $T\in\tau \Leftrightarrow \forall\lambda\in\Lambda, T\cap X_\lambda\in\tau_\lambda$

If (X_λ,τ_λ) is subspace of (X,τ) and X_λ is open in X for every $\lambda\in\Lambda$, (X,τ) is a free union of the subspaces (X_λ,τ_λ). But the spaces (X_λ,τ_λ) are not necessary to be a subspace of X. So we will give the definition of disjoint union topology and topological summed and investigate some of results about topological summed in this paper.

Let $\{(X_\lambda,\tau_\lambda)\}$ be a disjoint non-empty collection of topological spaces (X_λ,τ_λ) indexed by a set Λ. The disjoint union $X=\bigcup X_\lambda$, $\lambda\in\Lambda$ is a topological space with the following topology

$\tau=\{T\subseteq X: T\cap X_\lambda\in\tau_\lambda \text{for each } \lambda\in\Lambda\}$.

τ is a disjoint union topology and X is a topological summed of a disjoint non-empty collection of topological spaces X_λ. Hence, (X,τ) is a free union of (X_λ,τ_λ).

In this paper we understand $\{(X_\lambda,\tau_\lambda)\}_{\lambda\in\Lambda}$ is the disjoint non-empty collection of topological spaces indexed by a set Λ and (X,τ) is the topological summed of collection $\{(X_\lambda,\tau_\lambda)\}_{\lambda\in\Lambda}$.

Now we will give free union of a disjoint non-empty collection of topological spaces. Definitions, theorems and some results for topological summed have been obtained by using the known definitions and theorems for the topological spaces [1], [2], [3].

2. MAIN RESULTS

Our definition to be used in the future is the following.

Definition 2.1: For $(x_n)_{\mathbb{N}}$ sequence of X and $b\in X$, $(x)_n\to b \Leftrightarrow \exists!\lambda\in\Lambda, \exists N_k\in\mathbb{N}: n\geq N_k \Rightarrow x_n\in T^{(k)}$ for every $T^{(k)}\in\tau_\lambda$ containing b.

Theorem 2.1: If τ is the family in the sense of definition in Introduction, then (X,τ) is a topological space.

Proof: For each $\lambda\in\Lambda, \emptyset\cap X_\lambda=\emptyset\in\tau_\lambda \Rightarrow \emptyset\in\tau$, $X\cap X_\lambda=X_\lambda\in\tau_\lambda \Rightarrow X\in\tau_\lambda$.

For every finite subfamily $\{T_1,T_2,\dots,T_r\}\subseteq\tau$, we have $(\bigcap_{i=1}^{r}T_i)\cap X_\lambda=\bigcap_{i=1}^{r}(T_i\cap X_\lambda)$ for each $\lambda\in\Lambda$. Since $T_i\in\tau$, for every $\lambda\in\Lambda$ we have $T_i\cap X_\lambda\in\tau_\lambda$ and we know (X_λ,τ_λ) is topological space so $\bigcap_{i=1}^{r}(T_i\cap X_\lambda)\in\tau_\lambda$. Then $\bigcap_{i=1}^{r}(T_i\cap X_\lambda)=(\bigcap_{i=1}^{r}T_i)\cap X_\lambda\in\tau_\lambda$ and hence $\bigcap_{i=1}^{r}T_i\in\tau$.

For every subfamily $\{T_1,T_2,\dots\}\subseteq\tau$ and for each $\lambda\in\Lambda$ we have,

$(\bigcup_i T_i)\cap X_\lambda=\bigcup_i(T_i\cap X_\lambda)\in\tau_\lambda$

and hence $(\bigcup_i T_i)\in\tau$ ∎

Theorem 2.2: For each $\lambda\in\Lambda$, τ_λ is subfamily of τ.

Proof: Let $A\in\tau_\lambda$. Then we have $A\cap X_\lambda=A\in\tau_\lambda$. Also for $\lambda\neq\lambda', A\cap X_{\lambda'}=\emptyset\in\tau_\lambda$. Hence, we obtain $A\cap X_\lambda\in\tau_\lambda$, for each $\lambda\in\Lambda$ and so $A\in\tau$. ∎

Example 2.1: Let $\{(X_\lambda,\tau_\lambda)\}_{\lambda\in\Lambda}$ be a disjoint non-empty collection of topological spaces. It is clear that $\bigcup_{i\in I}T_i\in\tau$ for each $\lambda\in\Lambda, T_i\in\tau_\lambda$. Also by theorem 2.2 we obtain $\tau=\{\bigcup_{i\in I}T_i : \exists\lambda\in\Lambda, T_i\in\tau_\lambda\}$.

Suppose now that τ_λ is indiscrete topology. For every $\lambda\in\Lambda$, we have $X_\lambda\in\tau$. Hence, τ can not be indiscrete topology except $\Lambda=\{1\}$.

Theorem 2.3: τ is indiscrete iff $\Lambda=\{1\}$.

Proof: It is obtained from the above example easily. ∎

Theorem 2.4: $\boldsymbol{\tau}$ is discrete iff for each $\lambda\in\Lambda$, τ_λ is discrete.

Proof: If $\boldsymbol{\tau}$ is discrete, $\{x\}$ is element of $\boldsymbol{\tau}$, for each $x\in X$. Then $\{x\}\cap X_\lambda$ is element of τ_λ, for each $\lambda\in\Lambda$. As X_λ is discrete, $x\in X$ is element of X_λ for only one $\lambda\in\Lambda$, i.e., for $\lambda\neq\lambda', x\in X_\lambda \Rightarrow \{x\}\cap X_{\lambda'}=\emptyset$ and $\{x\}\cap X_\lambda=\{x\}\in\tau_\lambda$. So τ_λ contain all single point set, for each $\lambda\in\Lambda$. So τ_λ is discrete topology.

Conversely, if τ_λ is discrete, τ is discrete because of $\tau_\lambda\subseteq\tau$ ∎

Theorem 2.5: X_λ is subspace of X.

Proof: Because of $X=\bigcup_{\lambda\in\Lambda}X_\lambda$, X_λ is subset of X. However, for every $\lambda\in\Lambda$ and for $\tau'=\{X_\lambda\cap T: T\in\tau\}$ we must show that $\tau_\lambda=\tau'$.

$$T\in\tau' \Rightarrow \exists V\in\tau: T=X_\lambda\cap V \overset{V\in\tau}{\Longrightarrow} X_\lambda\cap V\in\tau_\lambda \Rightarrow T\in\tau_\lambda$$

$$U\in\tau_\lambda \overset{\tau_\lambda\subseteq\tau}{\Longrightarrow} U\in\tau \overset{U\subseteq X_\lambda}{\Longrightarrow} U=U\cap X_\lambda, U\in\tau \Rightarrow U\in\tau'$$

Then we obtain $\tau_\lambda=\tau'$ ∎

Theorem 2.6: If $A\cap X_i=\{x\}$ and $\{x\}\notin\tau_i$, for $x\in X$ and $i\in\Lambda$. A can't be neighborhood of x.

Proof: Since $\{(X_\lambda,\tau_\lambda)\}_{\lambda\in\Lambda}$ is the disjoint collection of topological spaces, X_i is the only subset such that $x\in X_i$. On the contrary, A is neighborhood of x such that $A\cap X_i=\{x\}$ and $\{x\}\notin\tau_i$. In this case, there is $U\in\tau$ such that $x\in U\subseteq A$. So $U\cap X_\lambda\in\tau_\lambda$, for each $\lambda\in\Lambda$, i.e., $U\cap X_i=$

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$\{x\} \in \tau_i$, for $i \in \Lambda$, contradiction. ■

Theorem 2.7: $A \subseteq X$ is open iff A is neighborhood of each $x \in A$.

Proof: Let $A \in \tau$, $x \in A$. Since we can write $x \in A \subseteq A$, A is neighborhood of the point x.

Conversely, if A is neighborhood of each $x \in A$, then there exist $U_x \in \tau$ such that $x \in U_x \subseteq A$. Hence, we have

$A = \bigcup_{x\in A}\{x\} \subseteq \bigcup_{x\in A} U_x \subseteq \bigcup_{x\in A} A = A$.

Since the set A is union of the open sets U_x, A is open set. ■

Theorem 2.8: For $x \in X_\lambda \subseteq X$ and $x \in A \subseteq X$, A is neighborhood of x in X such that $A \neq X_\lambda$ iff $A \cap X_\lambda$ is neighborhood of x in X_λ.

Proof: Let A be a neighborhood of x in X. So there is $U \in \tau$ such that $x \in U \subseteq A$. So that $A \cap X_\lambda$ is neighborhood of x in X_λ because of $x \in U \cap X_\lambda \subseteq A \cap X_\lambda$ and $U \cap X_\lambda \in \tau_\lambda$.

If $A \cap X_\lambda$ is neighborhood of x in X_λ, there is $U \in \tau_\lambda$ such that $x \in U \subseteq A \cap X_\lambda$. $U \in \tau$, because of $\tau_\lambda \subseteq \tau$. Also we know that $U \subseteq A \cap X_\lambda \subseteq A$ and so A is neighborhood of x in X. ■

Theorem 2.9: Let $A \subseteq X$. For $\bar{A}$ is closure of A, $\bar{A} = \bigcup_{\lambda\in\Lambda} \overline{A \cap X_\lambda}$.

Proof: Let x be a element of a set $\bar{A}$. Then $x \in K$ such that $K^c \in \tau$, for each $K \supseteq A$. Hence $x \in K \cap X_\lambda$ such that $(K \cap X_\lambda)^c \in \tau_\lambda$, for at least one $\lambda \in \Lambda$. So $x \in \bigcup_{\lambda\in\Lambda} \overline{A \cap X_\lambda}$.

Let x be a element of a set $\bigcup_{\lambda\in\Lambda} \overline{A \cap X_\lambda}$. Then $x \in \overline{A \cap X_\lambda}$, for at least one $\lambda \in \Lambda$. So $x \in \bar{A}$ because of $A \cap X_\lambda$. ■

Theorem 2.10: Let $A \subseteq X$. For A° is interior of A, $A^\circ = \bigcup_{\lambda\in\Lambda}(A \cap X_\lambda)^\circ$.

Proof: Let x be a element of a set A°. Then $x \in G$ such that $G \in \tau$, for at least one $G \subseteq A$. Therefore $x \in G \cap X_\lambda$, for at least one $\lambda \in \Lambda$ and $G \subseteq A$. So $x \in G \cap X_\lambda \subseteq A \cap X_\lambda$ such that $x \in G \cap X_\lambda$. Thus $x \in \bigcup_{\lambda\in\Lambda}(A \cap X_\lambda)^\circ$ and so $A^\circ \subseteq \bigcup_{\lambda\in\Lambda}(A \cap X_\lambda)^\circ$.

Let x be a element of a set $\bigcup_{\lambda\in\Lambda}(A \cap X_\lambda)^\circ$. Then $x \in (A \cap X_\lambda)^\circ$, for at least one $\lambda \in \Lambda$. Thus $x \in A^\circ$ because of $A \cap X_\lambda \subseteq A$. So $\bigcup_{\lambda\in\Lambda}(A \cap X_\lambda)^\circ \subseteq A^\circ$. ■

Theorem 2.11: The family $\boldsymbol{\mathcal{B}} = \{T \subseteq X : \exists\lambda \in \Lambda,\ T \in \tau_\lambda\}$ is the base of (X, τ) topological space.

Proof: i) We know that for every $\lambda \in \Lambda$, $X_\lambda \in \tau_\lambda$ and $\bigcup_{\lambda\in\Lambda} X_\lambda = X$

ii) Let $B_i, B_j \in \mathcal{B}$.So for at least one $i, j \in \Lambda$, $B_i \in \tau_i, B_j \in \tau_j$. For $i \neq j$, the condition is obvious because of $B_i \cap B_j = \emptyset$. Let $i = j$ and $B_i \cap B_j \neq \emptyset$. We know that if $B_i, B_j \in \tau_i$, $B_i \cap B_j \in \tau_i$. So for $B_{ij} = B_i \cap B_j$, and $x \in B_i \cap B_j$, there exist $B_{ij} \in \mathcal{B}$ such that $x \in B_{ij} \subseteq B_i \cap B_j$. ■

Theorem 2.12: Let the family $\mathcal{B}_\lambda$ be a base of X_λ topological space for $\lambda \in \Lambda$. In this case, $\mathfrak{B} = \{T : \exists\lambda \in \Lambda,\ T \in \mathcal{B}_\lambda\}$ is the base of (X, τ) topological space.

Proof: i) Since $\mathcal{B}_\lambda$ is the base of X_λ topological space, it is obvious that $\bigcup_{T\in\mathcal{B}_\lambda} T = X$.

ii) For every $\lambda \in \Lambda$, it is obvious that each $\mathcal{B}_\lambda$ is discrete. For $i, j \in \Lambda$, let $T_i \in \mathcal{B}_i, T_j \in \mathcal{B}_j$ and for $i = j$, $T_i \cap T_j \neq \emptyset$. As $\mathcal{B}_i$ is a base of X_i , for every $x \in T_i \cap T_j$, there exist $T_{ij} \in \mathcal{B}_{ij}$ such that $x \in T_{ij} \subseteq T_i \cap T_j$. ■

Theorem 2.13: The convergent sequence of X_λ converges in X also.

Proof: Let (x_n) be such a sequence of X_λ that converges the point $b \in X_\lambda$ in X_λ. (x_n) also is a sequence of X because of $X_\lambda \subseteq X$. From the definition of convergence we have;

For $\forall T^\lambda \in \tau_\lambda(b \in T^\lambda), \exists n_0^\lambda \in \mathbb{N} : n \geq n_0^\lambda \Rightarrow x_n \in T^\lambda$.

On the other hand, for $\forall T \in \tau(b \in T)$ we know that $T = T^\lambda$ or $T \supseteq T^\lambda$. So for $n_0 = n_0^\lambda$, if $n \geq n_0$, $x_n \in T$. Then (x_n) converges in X. ■

We note that the sequence of X need not be converging in X_λ.

Let us define the set $Z_G = \{n \in \mathbb{N} : x_n \notin G\}$ for (x_n) is a sequence of X and $G \in \tau$. Now, we will talk about a different approach for convergence with maksZ_G. Also, we will take maks$Z_G = 1$ while $Z_G = \emptyset$.

Theorem 2.14: Let (x_n) be a sequence of X. (x_n) converges the point $b \in X$ iff there exist a maksZ_G for each $G \in \tau(b \in G)$.

Proof: Let $N_k = \text{maks}\{n_G : n_G = \text{maks}Z_G,\ G \in \tau, b \in G\}$. Then (x_n) converges the point $b \in X$ In the meanings given in the definition 2.1. So (x_n) converges the point $b \in X$ in the meanings given in the Theorem 2.14 also. ■

Theorem 2.15: Let (x_n) be a sequence of X, $\lambda, \mu \in \Lambda$, $b \in X_\lambda$ and $b' \in X_\mu$. If b and b' is limit point for (x_n), $\lambda = \mu$.

Proof: On the contrary, let $\lambda \neq \mu$. Then $X_\lambda \cap X_\mu = \emptyset$. Since (x_n) converges the point $b \in X_\lambda$, for $n \geq n_0$, there exist $n_0 \in \mathbb{N}$ such that $x_n \in X_\lambda$. On the other hand, since (x_n) converges the point $b' \in X_\mu$, for $n \geq m_0$, there exist $m_0 \in \mathbb{N}$ such that $x_n \in X_\mu$. Let $p_0 = maks\{n_0, m_0\}$. For $n \geq p_0$, $x_n \in X_\lambda \cap X_\mu$, contradiction. ■

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